SOME Q-FEJÉR INEQUALIRIES FOR $\left(\mathcal{M}_\phi,\mathcal{M}_\psi\right)$-CONVEX FUNCTIONS

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Ngày nhận bài: 01/11/2023; Ngày phản biện thông qua: 27/12/2023; Ngày duyệt đăng: 28/12/2023

ABSTRACT

In this paper, we consider a class of generalized convex functions, which are defined according to a pair of quasi-arithmetic means and called $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex functions and establish some q-Fejér inequalities for such a function class.

Keywords: *Convex function, Hermite-Hadamard inequalitiy, Fejér inequality, q-intergral inequality, q-calculus.*

1. INTRODUCTION

The Hermite-Hadamard inequality was first introduced in 1883 by Hermite (Hermite, 1883) and 10 years later by Hadamard (Hadamard, 1893): Let $f:[a,b]\to\mathbb{R}$ be a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \quad (1)$$

The weighted form of the Hermite-Hadamard inequality was given by Fejér (Fejér, 1996): If $f:[a,b]\to\mathbb{R}$ is a convex function, $g:[a,b]\to[0,\infty)$ is an integrable function with $\int_a^b g(x)dx>0$ and symmetry to $\frac{a+b}{2}$,i.e., $g(x)=g(a+b-x)$ for all $x\in[a,b]$, then

$$f\left(\frac{a+b}{2}\right)\le\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}\le\frac{f(a)+f(b)}{2}. \quad (2)$$

The concept of q-calculus was introduced by Jackson in 1910 (Jackson, 1910), then widely applied in other fields such as number theory, combinatorics, hypergeometric functions, orthogonal polynomials, quantum theory, mechanics and is considered a bridge between mathematics and physics (see Ernst, 2012, Kac & Cheung, 2022). In articles Tariboon & Ntouyas, 2013, 2014, the concept of q-derivative and q-integral on a finite $[a,b]\subset\mathbb{R}$ and established a number of versions of integral inequalities for the concept of q-integral.

Alp and co-authors (Apl & et at., 2018) established the q-Hermite-Hadamard inequality for left q-integrals: If $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{q\alpha+b}{1+q}\right)\le\frac{1}{b-a}\int_a^b f(x)_\alpha d_qx\le\frac{qf(a)+f(b)}{1+1}. \quad (3)$$

Bermudo and colleagues in 2020 introduced the concept of right q-integral and established the q-Hermite-Hadamard inequality for convex functions $f:[a,b]\to\mathbb{R}$ as follows

$$f\left(\frac{a+qb}{1+q}\right)\le\frac{1}{b-a}\int_a^b f(x)^b d_qx\le\frac{f(a)+qf(b)}{1+q}. \quad (4)$$

Inequalities (3) and (4) have recently received extensive research attention, see for example works Liu & Hefeng, 2016, Noor & et al., 2015.

Most recently, Ali and co-authors in 2023 proposed another version of the q-Hermite-Hadamard inequality involving left q-integral and right q-integral as follows: If $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)_\alpha d_qx+\int_{\frac{a+b}{2}}^b f(x)^b d_qx\right]\le\frac{f(a)+f(b)}{2}. \quad (5)$$

Inspired by the studies mentioned above, in this paper, we establish some q-Fejér type inequalities for the class of $\left(\mathcal{M}_\phi,\mathcal{M}_\psi\right)$-convex functions. Our results are extensions and refinements of recent results for the q-Hermite-Hadamard inequality.

2. RESEARCH CONTENTS AND METHODS

2.1. Research contents

- $\left(\mathcal{M}_\phi,\mathcal{M}_\psi\right)$-convex function.
- The q-Fejér inequality for the class of $\left(\mathcal{M}_\phi,\mathcal{M}_\psi\right)$-convex functions.

2.2. Research methods

- Theoretical mathematical research includes analysis, comparison, contrast, generalization, and specialization to predict and introduce new inequalities;
- Using estimates and assessments based on convex function theory;
- Incorporates several new methods and

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techniques recently developed by us when constructing Fejér-type inequalities for $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex functions.

3. RESULTS AND DISCUSSIONS

First, we recall the definition of a $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex function and its basic properties. In this article, symbols I and J are real number intervals, $\phi: I\to\mathbb{R}$ and $\psi: J\to\mathbb{R}$ are strictly monotone and continuous functions. Use the pair of quasi-arithmetic mean $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$, where

$$\mathcal{M}_\psi(a,b;\alpha)=\phi^{-1}(\alpha\phi(a)+(1-\alpha)\phi(b))$$

Aummann (Aummann, 1933) introduced the concept of $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex function as follows.

Definition 1 (Aummann, 1933). $f: I\to J$ is called $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex function if

$$f\left(\mathcal{M}(a,b;\alpha)\right)\le\mathcal{M}_\psi\left(f(a),f(b);\alpha\right) \quad (6)$$

for all $a,b\in I$ and $\alpha\in[0;1]$.

In the case where inequality (6) is satisfied with $\phi(x)=x$, we say f is $\mathcal{M}_\psi$ -convex, and if f satisfies inequality (6) with $\phi(x)=x$ and $\psi(x)$, then f is a convex function.

Note, if ψ is an increasing function then $f: I\to J$ is $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex if and only if $\psi\circ f\circ\phi^{-1}$ is convex on $\phi(I)$. And if ψ is a decreasing function, then $f: I\to J$ is $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex if and only if $\psi\circ f\circ\phi^{-1}$ is a concave function on $\phi(I)$.

Lemma 2. (*Niculescu & Persson, 2006, Lemma A.22) If ψ is increasing on J, then $f: I\to J$ is $(\mathcal{M}_\phi,\mathcal{M}_\psi)$-convex function if and only if $g(x)=\psi\circ f\circ\phi^{-1}(x)$ is convex on $\phi(I)$.*

Next is some knowledge about q-derivatives and q-integrals where q is always understood as a real number in the range $(0,1)$.

Definition 3. (Tariboon & Ntouyas, 2013) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then q-the left derivative of f at $x\in[a,b]$ is defined as follows

$$_aD_qf(x)=\begin{cases}\dfrac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, & x\ne a\\ \lim_{x\to a}\,{}_aD_qf(x) & ,x=a\end{cases}$$

A function f is called q-differentiable on $[a,b]$ if $_aD_qf(x)$ exists for all $x\in[a,b]$.

Definition 4. (Tariboon & Ntouyas, 2013) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then q-left integral of f at $x\in[a,b]$ is defined as follows

$$\int_a^x f(t)_a d_q(t)=(1-q)(x-a)\sum_{n=0}^{\infty}q^n f(q^n x+(1-q^n)a).$$

A function f is called q- integrable on $[a,b]$ if $\int_a^x f(x)_a d_q(t)$ exists for all $x\in[a,b]$.

In particular, if $a=0$ then we get the Jackson q-integral (Jackson, 1910)

$$\int_0^x f(t)_0 d_q(t)=\int_0^x f(t)d_q(t)=(1-q)x\sum_{n=0}^{\infty}q^n f\left(q^n x\right).$$

Comment 5. (Bermudo & et al., 2020, Tariboon & Ntouyas, 2013, 2014) Let $f:[a,b]\to\mathbb{R}$ be a continuous function, then

1. $_aD_q\int_a^x f(x)_a d_q(t)=f(x)-f(a).$
2. $\int_c^x {}_aD_qf(x)d_q(t)=f(x)-f(c)$ for all $c\in(a,x)$.
3. $\int_a^x[\alpha f(x)+\beta g(x)]_a d_q(t)=\alpha\int_a^x f(x)_a d_q(t)+\beta\int_a^x g(x)_a d_q(t).$
4. If g is a continuous function on $[a,b]$ and $f(t)\le g(t)$ for all $t\in[a,b]$ then $\int_a^x f(t)_a d_q(t)\le\int_a^x g(t)_a d_q(t).$

On the other hand, Bermudo and colleagues in 2020 introduced the concepts of right q-derivative and right q-integral as follows.

Definition 6. (Bermudo & et al., 2020) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then, q-the left derivative of f at $x\in[a,b]$ is defined as follows

$$^bD_qf(x)=\begin{cases}\dfrac{f(qx+(1-q)b)-f(x)}{(1-q)(b-x)}, & x\ne b\\ \lim_{x\to b}\,{}^bD_qf(x) & ,x=b\end{cases}$$

A function f is called q- differentiable on $[a,b]$ if $_aD_qf(x)$ exists for all $x\in[a,b]$.

Definition 7. (Bermudo & et al., 2020) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then q-right integral of f at $x\in[a,b]$ is defined as follows

$$\int_x^b f(t)^b d_q(t)=(1-q)(b-x)\sum_{n=0}^{\infty}q^n f\left(q^n x+\left(1-q^n\right)b\right)$$

A function f is called q-integrable right on $[a,b]$ if $\int_x^b f(x)^b d_q(t)$ exists for all $x\in[a,b]$.

From Remark 5, we can also obtain similar properties for the right q-integral.

Next, we will establish and prove the q-Hermite-Hadamard inequality and the q-Fejér inequality for the $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex function.

In this article, we always assume $f: I\to J$ is a $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -convex function; $a,b\in I$ with $a<b$; $\alpha\in(0;1)$; $q\in(0;1)$; $w:[0,1]\to[0,\infty)$

is a q-integral function and satisfies the condition $\int_0 w(t)d_qt > 0$ for all $s \in (ü \quad]$. Symbol

$$\mathcal{L}(t) = \mathcal{M}_\phi\left(a, \mathcal{M}_\phi(a,b;\alpha);t\right)$$

and

$$\mathcal{R}(t) = \mathcal{M}_\phi\left(b, \mathcal{M}_\phi(a,b;\alpha);t\right)$$

with $t \in [0,1]$.

Lemma 8. (Duc & et al., 2020) *Let* $\mathcal{F}, \mathcal{G}: [0,1] \to \mathbb{R}$ *be functions respectively defined by*

$$\mathcal{F}(t) = \mathcal{M}_\psi\left(f \circ \mathcal{L}(t), f \circ \mathcal{R}(t);\alpha\right)$$

and

$$\mathcal{G}(t) = \mathcal{M}_\psi\left(\mathcal{F}(1), \mathcal{F}(0);t\right).$$

Then, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{M}_\psi$-convex, increasing functions on $[0,1]$ and

$$\mathcal{F}(0) = \mathcal{G}(0) = f\left(\mathcal{M}_\phi(a,b;\alpha)\right)$$
$$\mathcal{F}(t) \le \mathcal{G}(t), \quad t \in [0,1], \tag{7}$$
$$\mathcal{F}(1) = \mathcal{G}(1) = \mathcal{M}_\phi\left(f(a), f(b);\alpha\right).$$

Theorem 9 (q-Hermite-Hadamard inequality). *Let* $f: I \to J$ *be a* $(\mathcal{M}_\phi, \mathcal{M}_\psi)$*-convex function. Then*

$$\begin{aligned} & f\left(\mathcal{M}_\phi(a,b;\alpha)\right) \\ & \le \frac{(1-\alpha)}{\phi(b)-\phi(a)} \int_a^{\mathcal{M}_\phi(a,b;\alpha)} \psi \circ f(x)\, {}_a d_q\phi(x) \\ & + \frac{\alpha}{\phi(b)-\phi(a)} \int_{\mathcal{M}_\phi(a,b;\alpha)}^b \psi \circ f(x)\, {}^b d_q\phi(x) \\ & \le \frac{1}{1+q}\psi\left(\mathcal{M}_\phi(f(a), f(b);\alpha)\right) \\ & \quad + \frac{q}{1+q}\psi \circ f\left(\mathcal{M}_\phi(a,b;\alpha)\right). \end{aligned} \tag{8}$$

Proof. For all $t \in [0,1]$

$$\begin{aligned} \psi \circ \mathcal{F}(0) \le \psi \circ \mathcal{F}(t) &= \alpha \psi \circ f \circ \phi^{-1}\left(A(t)\right) \\ &+ (1-\alpha)\psi \circ f \circ \phi^{-1}\left(B(t)\right) \end{aligned} \tag{9}$$

where

$$A(t) = t\phi(a) + (1-t)\left(\alpha\phi(a) + (1-\alpha)\phi(b)\right)$$

and

$$B(t) = t\phi(b) + (1-t)\left(\alpha\phi(a) + (1-\alpha)\phi(b)\right).$$

Taking q-integrate both sides of (9) we get

$$\begin{aligned} \psi \circ f\left(\mathcal{M}_\phi(a,b;\alpha)\right) &\le \alpha \int_0^1 \psi \circ f \circ \phi^{-1}(A(t))d_qt \\ & + (1-\alpha)\int_0^1 \psi \circ f \circ \phi^{-1}(B(t))d_qt \\ & = \frac{(1-\alpha)}{\phi(b)-\phi(a)} \int_a^{\mathcal{M}_\phi(a,b;\alpha)} \psi \circ f(x)\, {}_a d_q\phi(x) \\ & + \frac{\alpha}{\phi(b)-\phi(a)} \int_{\mathcal{M}_\phi(a,b;\alpha)}^b \psi \circ f(x)\, {}^b d_q\phi(x). \end{aligned}$$

On the other hand

$$\begin{aligned} \psi \circ \mathcal{F}(t) &\le t\psi\left(\mathcal{M}_\phi\left(f(a), f(b);\alpha\right)\right) \\ & + (1-t)\psi \circ f\left(\mathcal{M}_\phi(a,b;\alpha)\right). \end{aligned}$$

Taking q-integrate both sides we get

$$\begin{aligned} \int_0^1 \psi \circ \mathcal{F}(t)d_qt &\le \psi\left(\mathcal{M}_\phi(f(a), f(b);\alpha)\right)\int_0^1 t d_qt \\ & + \psi \circ f\left(\mathcal{M}_\phi(a,b;\alpha)\right)\int_0^1 (1-t)d_qt \\ & = \frac{1}{1+q}\psi\left(\mathcal{M}_\phi(f(a), f(b);\alpha)\right) \\ & + \frac{q}{1+q}\psi \circ f\left(\mathcal{M}_\phi(a,b;\alpha)\right). \end{aligned}$$

The theorem has been proven.

Comment 10. In case f is a convex function $\psi(x) = \phi(x) = x$, inequality (8) becomes q-Hermite-Hadamard inequality (5). By choosing special functions ψ and ϕ, we will get q-Hermite-Hadamard inequalities for generalized convex functions such as log-convex function, harmonic convex function, kernel convex function, harmonic log-convex function, p-convex function,…

Theorem 11. *With* $s \in (0,1]$*, we set*

$$\mathcal{L}(s) = \psi^{-1}\left(\frac{\int_0^s \psi \circ \mathcal{F}(t)w(t)d_qt}{\int_0^s w(t)d_qt}\right)$$

and

$$\beta(s) = \frac{\int_0^s t w(t)d_qt}{\int_0^s w(t)d_qt}.$$

Then $\mathcal{F} \circ \beta$, $\mathcal{L}$ and $\mathcal{G} \circ \beta$ are increasing functions on $(0,1]$ and satisfied

$$\lim_{s\to 0^+} \mathcal{F} \circ \beta(s) = \lim_{s\to 0^+} \mathcal{I}(s) = \lim_{s\to 0^+} \mathcal{G} \circ \beta(s) = \mathcal{G}(0),$$

$$\mathcal{F} \circ \beta(s) \le \mathcal{I}(s) \le \mathcal{G} \circ \beta(s), \quad s \in (0,1]. \tag{10}$$

To prove the above theorem, we need the following result.

Lemma 12. *Let* $P: [0,1] \to \mathbb{R}$ *be a increasing, continuous function. With* $s \in (0,1]$*, we set*

$$P_1(s) = \frac{\int_0^s P(t)w(t)d_qt}{\int_0^s w(t)d_qt}.$$

Then P_1 *is a increasing function on* $(0,1]$ *and*

$$\lim_{s\to 0^+} P_1(s) = P(0) \le P_1(s) \le P(s),\ s \in (0,1]. \tag{11}$$

Proof. Proof similar to Duc & et al. 2020 Lemma 2.4.

Now we prove Theorem 11.

Proof Theorem 11. Since ψ is strictly monotonic, we need to consider two cases of ψ.

Suppose first that ψ increases strictly on *J*. Since ψ is continuous on *J*, ψ^{-1} is continuous and strictly increasing on $\psi(J)$.

Applying Lemma 8 and Lemma 12 to $P=\psi\circ\mathcal{F}$, $\psi\circ\mathcal{L}$ increases on $(0,1]$ with

$$\lim_{s\to0^+}\psi\circ\mathcal{L}(s)=\psi\circ\mathcal{F}(0).$$

Since ψ^{-1} is continuous and strictly increasing on on $\psi(J)$, $\mathcal{L}$ increases on $(0,1]$ and

$$\lim_{s\to0^+}\mathcal{L}(s)=\psi\circ\mathcal{F}(0).$$

Again according to Lemma 12, we have β increasing on $(0,1]$ with

$$\lim_{s\to0^+}\beta(s)=0\le\beta(s)\le s,\qquad s\in(0,1].$$

Therefore $\mathcal{F}\circ\beta$ and $\mathcal{G}\circ\beta$ determine, increase on $(0,1]$ and

$$\lim_{s\to0^+}\mathcal{F}\circ\beta(s)=\lim_{s\to0^+}\mathcal{G}\circ\beta(s)=\mathcal{G}(0).$$

Next, we prove the inequalities in (10). Fixed $s\in(0,1]$. Applying Jensen's inequality (see Pečarić & et al., 1992, Chapter 2) to the convex function $\psi\circ\mathcal{F}$ on the interval $[0,s]$, we get

Because $\mathcal{F}(t)\le\mathcal{G}(t)$, so

$$\psi\circ\mathcal{F}\left(\frac{\int_0^s tw(t)d_qt}{\int_0^s w(t)d_qt}\right)\le\frac{\int_0^s\psi\circ\mathcal{F}(t)w(t)d_qt}{\int_0^s w(t)d_qt}.$$

Inferred that

$$\mathcal{F}\circ\beta(s)\le\mathcal{L}(s).$$

Due to $\mathcal{F}(t)\le\mathcal{G}(t)$, $t\in[ü\ ]$, the continuity of the functions $\mathcal{F}(t)$ and $\mathcal{G}(t)$ on $[0,1]$ along with the monotonicity of the *q*-integral, we have

$$\frac{\int_0^s\psi\circ\mathcal{F}(t)w(t)d_qt}{\int_0^s w(t)d_qt}\le\frac{\int_0^s\psi\circ\mathcal{G}(t)w(t)d_qt}{\int_0^s w(t)d_qt}=\psi\circ\mathcal{G}\circ\beta(s).$$

Because ψ^{-1} increases

$$\mathcal{L}(s)\le\mathcal{G}\circ\beta(s).$$

The theorem is proven similarly for the case where ψ is a decreasing function.

From Theorem 11, we can establish a number of Fejér-type inequalities for $(\mathcal{M}_\phi,\mathcal{M}_\psi)$-convex functions by choosing different w functions. For example, we choose

$$w(t)=(1-\alpha)g\circ\mathcal{L}(t)+\alpha g\circ\mathcal{R}(t),\quad t\in[0,1],$$

where $g:[a,b]\to[0,\infty)$ is chosen satisfactorily

$$\frac{1-\alpha}{\alpha}g\circ\mathcal{L}(t)=\frac{\alpha}{1-\alpha}g\circ\mathcal{R}(t),\ t\in[0,s]\qquad(12)$$

Note that when $\alpha=1/2$ and $\phi(x)=x$, Assumption (12) reduces to the assumption that g is symmetric about $(a+b)/2$.

$$\begin{aligned}\int_0^s w(t)d_qt&=(1-\alpha)\int_0^s g\circ\mathcal{L}(t)d_qt+\alpha\int_0^s g\circ\mathcal{R}(t)d_qt\\&=\frac{1}{\phi(b)-\phi(a)}\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}g(x)_a d_q\phi(x)\\&\quad+\frac{1}{\phi(b)-\phi(a)}\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}g(x)^b d_q\phi(x)\end{aligned}$$

and

$$\begin{aligned}&\int_0^s\psi\circ\mathcal{F}(t)w(t)d_qt\\&=(1-\alpha)\int_0^s(\psi\circ f\circ\mathcal{L}(t))g\circ\mathcal{L}(t)d_qt\\&\quad+\alpha\int_0^s(\psi\circ f\circ\mathcal{R}(t))g\circ\mathcal{R}(t)d_qt\\&=\frac{1}{\phi(b)-\phi(a)}\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}(\psi\circ f(x))g(x)_a d_q\phi(x)\\&\quad+\frac{1}{\phi(b)-\phi(a)}\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}(\psi\circ f(x))g(x)^b d_q\phi(x)\end{aligned}$$

And so

$$\begin{aligned}&\mathcal{L}(s)\\&=\psi^{-1}\left[\frac{\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}(\psi\circ f(x))g(x)_a d_q\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}g(x)_a d_q\phi(x)+\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}g(x)^b d_q\phi(x)}\right.\\&\left.+\frac{\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}(\psi\circ f(x))g(x)^b d_q\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}g(x)_a d_q\phi(x)+\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}g(x)^b d_q\phi(x)}\right].\end{aligned}$$

Together with Theorem 11, we have the following result.

Corollary 13 (*q*-Fejér inequality). *Let* $f:I\to J$ *be a* $(\mathcal{M}_\phi,\mathcal{M}_\psi)$*-convex function. Suppose* $g:[a,b]\to[0,\infty)$ *is a q-integral function, with*

$$\int_0^s g\circ\mathcal{L}(t)d_qt>0$$

for all $s\in(0,1]$ *and satisfy* (12). *Then, for all* $s\in(0,1]$*, we have*

$$\begin{aligned}&f(\mathcal{M}_\phi(a,b;\alpha))\\&\le\mathcal{F}\left(\frac{\int_0^s tg_1\circ\mathcal{L}(t)d_qt}{\int_0^s g_1\circ\mathcal{L}(t)d_qt}\right)\\&\le\psi^{-1}\left[\frac{\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}(\psi\circ f(x))g(x)_a d_q\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)}g(x)_a d_q\phi(x)+\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}g(x)^b d_q\phi(x)}\right.\end{aligned}$$

$$+\frac{\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)}(\psi\circ f(x))g(x)^b d_q\phi(x)}{\int_{\mathcal{L}(s)}^{\mathcal{M}_\phi(a,b;\alpha)} g(x)_a d_q\phi(x)+\int_{\mathcal{M}_\phi(a,b;\alpha)}^{\mathcal{R}(s)} g(x)^b d_q\phi(x)}\Bigg]$$

$$\le \mathcal{G}\left(\frac{\int_0^s tg\circ\mathcal{L}(t)d_qt}{\int_0^s g\circ\mathcal{L}(t)d_qt}\right)$$

$$\le \psi^{-1}(\alpha\psi\circ f(a)+(1-\alpha)\psi\circ f(b)). \qquad (13)$$

Comment 14.

1. In (13), if given $p\to 1$ then we get the Fejér inequality for the $(\mathcal{M}_\phi,\mathcal{M}_\psi)$-convex function established in Duc & et al., 2020.

2. By choosing special functions ψ and ϕ, we will get q-Fejér inequalities for generalized convex functions, such as log-convex function, harmonic convex function, kernel convex function, harmonic log-convex function, p-convex function.

3. In addition, inequality (13) is also an extension and smoothing of inequalities (5), (8). Indeed, choose $\alpha=1/2$, $g=1$ and $\psi(x)=\phi(x)=x$. Then, inequality (13) follows

$$f\left(\frac{a+b}{2}\right)$$

$$\le\frac{f\left(\frac{(2q+3)a+(2q+1)b}{4(1+q)}\right)+f\left(\frac{(2q+1)a+(2q+3)b}{4(1+q)}\right)}{2}$$

$$\le\frac{2}{b-a}\left[\int_{\frac{(q+2)a+b}{2(1+q)}}^{\frac{a+b}{2}} f(x)_a d_qx+\int_{\frac{a+b}{2}}^{\frac{a+(q+2)b}{2(1+q)}} f(x)^b d_qx\right]$$

$$\le\frac{f\left(\frac{(q+2)a+b}{2(1+q)}\right)+f\left(\frac{a+(q+2)b}{2(1+q)}\right)}{2}$$

$$\le\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)_a d_qx+\int_{\frac{a+b}{2}}^{b} f(x)^b d_qx\right]$$

$$\le\frac{1}{1+q}\frac{f(a)+f(b)}{2}+\frac{q}{1+q}f\left(\frac{a+b}{2}\right)$$

$$\le\frac{f(a)+f(b)}{2}.$$

4. CONCLUSION

In the article, we have established and proven the q-Hermite-Hadamard inequality and q-Fejér inequality for the class of $\mathcal{M}_\phi,\mathcal{M}_\psi)$-convex functions. The new inequalities are extended, smoothed results for the q-Hermite-Hadamard inequality for the class of convex functions. The new techniques in estimation and evaluation used in the article can be applied for further research in the field of q-integral inequalities related to other classes of generalized convex functions.

MỘT SỐ BẤT ĐẲNG THỨC Q-FEJÉR CHO HÀM $(\mathcal{M}_\phi,\mathcal{M}_\psi)$-LỒI

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Received Date: 01/11/2023; Revised Date: 27/12/2023; Accepted for Publication: 28/12/2023

TÓM TẮT

Trong bài báo này, chúng tôi xem xét một lớp hàm lồi mở rộng liên quan đến một cặp tựa trung bình số học, được gọi là hàm $(\mathcal{M}_\phi,\mathcal{M}_\psi)$ -lồi từ đó thiết lập một số bất đẳng thức q-Fejér cho lớp hàm lồi này.

Từ khóa: *Hàm lồi, Bất đẳng thức Hemiter-Hadamard, Bất đẳng thức Fejér, Bất đẳng thức q-tích phân, q-giải tích*

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