NEW TYPES OF WEIGHTED HÖLDER MEANS AND CONVEXITY

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ABSTRACT

In this paper, based on a new concept of weighted Hölder means, we propose a new notion of convexity for real-valued functions named generalized convex function of Hölder type and investigate their properties.

Keywords: *Generalized convex function; Weighted Hölder means; Hölder convexity.*

1. INTRODUCTION

A crucial notion of convex analysis is convexity of sets and functions. Recall that a function $f : C \subset \mathbb{R} \to \mathbb{R}$, defined on a convex set C, is convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad (1)$$

holds for all $x, y \in C$ and $\lambda \in [0,1]$, and this concept dates back to Jensen's work in (Aleman, 1985). Since then, the theory of convex functions has been extensively studied and become an interesting topic in mathematics. Nonetheless the theory of convex functions is an important theory per se that touches almost all branches of mathematics. This theory has found significant roles in the various areas of mathematics and in almost all aspects of mathematical programming. In recent years, numerous extensions of convex functions have been proposed and studied for applications; see for instances (Aleman, 1985; Anderson et al., 2007; Dragomir, 2001; Dragomir & Pearce, 2000; Fang & Shi, 2014; Mohan & Neogy, 1995; Varošanec, 2007; Zabandan et al., 2012; Zhang et al., 2009) for scalar convexity and (Ando & Hiai, 2011; Hoa & Khue, 2018; Hoa et al., 2018)for operator convexity. From the view point of mentioned papers, general idea to derive generalized convex functions is replacing the weighted arithmetic mean by more general weighted means. This means that a generalized convex function can be defined with the requirement that

$$f(\mathrm{M}_1(x, y)) \leq \mathrm{M}_2\big(f(x), f(y)\big) \quad (2)$$

holds for all *x, y* taking values in the domain of f, where M_i stands for a selected-reader generalized mean; see (Bullen, 2003; Toader & Toader, 2007) for a survey. Each choice of means will lead to a kind of generalized convex functions.

The goal of this paper is to propose and study a new class of convex functions in the framework of (2), where involved means are real-order Hölder type weighted means. The motivation behind studying such functions comes from the reason that we would introduce a new class of convex functions that generalizes and covers recent results of real-valued one-variable convex functions and then find applications.

2. RESEARCH CONTENTS AND METHODS

2.1. Research contents

- The generalized means of Hölder type for any two real numbers and its properties.
- The generalized convex functions of Hölder type.

2.2. Research methods

Theoretical mathematical research includes analysis, comparison, contrast, generalization, and specialization to predict and introduce new results.

3. RESULTS AND DISCUSSIONS

3.1. Weighted means of Hölder type

We start our discussion with Hölder means of finite set of positive numbers which can be found in various standard texts on means (see, e.g. (Bullen, 2003) and the references therein).

Let r be a real number and x_1, x_2 be positive real numbers. It is well-known that the weighted Hölder mean of order r of x_1 and x_2 with respect to the weight $\mathbf{w} = (w_1, w_2)$ with $w_1 \geq 0, w_2 \geq 0, w_1 + w_2 > 0,$ is defined by

$$\mathrm{M}^{[r]}(x_1, x_2; w_1, w_2) = \begin{cases} \left(\dfrac{w_1}{|\mathbf{w}|} x_1^r + \dfrac{w_2}{|\mathbf{w}|} x_2^r\right)^{\frac{1}{r}} & \text{if } r \neq 0 \\ \left(x_1^{w_1} x_2^{w_2}\right)^{\frac{1}{|\mathbf{w}|}} & \text{if } r = 0 \end{cases} \quad (3)$$

where $|\mathbf{w}| = w_1 + w_2$. Note that

$$\lim_{r \to 0} \mathrm{M}^{[r]}(x_1, x_2; w_1, w_2) = \mathrm{M}^{[0]}(x_1, x_2; w_1, w_2). \quad (4)$$

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Based on geometrical ideas we propose a general notion of the weighted Hölder means with a real order r of two real numbers which are not necessary positive.

Let $\mathcal{R}_+^2$ be the set of all $(w_1, w_2) \in \mathbb{R}_+^2$ such that $w_1 + w_2 = 1$. Consider the function $\mathcal{M} : \mathbb{R} \times \mathbb{R}^2 \times \mathcal{R}_+^2 \to \mathbb{R}, (r, \mathbf{x}, \mathbf{w}) \mapsto \mathcal{M}(r, \mathbf{x}, \mathbf{w})$ that is defined as

$$\mathcal{M}(r,\mathbf{x},\mathbf{w}) :=$$

$$(5)\begin{cases} (w_1 x_1^r + w_2 x_2^r)^{\frac{1}{r}} & \text{if } x_1 > 0, x_2 > 0, \\ -(w_1(-x_1)^r + w_2(-x_2)^r)^{\frac{1}{r}} & \text{if } x_1 < 0, x_2 < 0, \\ \lim\limits_{\tilde{x}_1 \to 0^+} \mathcal{M}(r, \tilde{x}_1, x_2, w_1, w_2) & \text{if } x_1 = 0, x_2 > 0, \\ \lim\limits_{\tilde{x}_2 \to 0^+} \mathcal{M}(r, x_1, \tilde{x}_2, w_1, w_2) & \text{if } x_1 > 0, x_2 = 0, \\ 0 & \text{if } x_1 = 0, x_2 = 0, \\ \lim\limits_{\tilde{x}_1 \to 0^-} \mathcal{M}(r, \tilde{x}_1, x_2, w_1, w_2) & \text{if } x_1 = 0, x_2 < 0, \\ \lim\limits_{\tilde{x}_2 \to 0^-} \mathcal{M}(r, x_1, \tilde{x}_2, w_1, w_2) & \text{if } x_1 < 0, x_2 = 0, \\ 2x_2 - (w_2 x_2^r + w_1(2x_2 - x_1)^r)^{\frac{1}{r}} & \text{if } x_1 < 0 < x_2, \\ 2x_1 - (w_1 x_1^r + w_2(2x_1 - x_2)^r)^{\frac{1}{r}} & \text{if } x_2 < 0 < x_1, \end{cases}$$

for $r \neq 0$ and

$$\mathcal{M}(0,\mathbf{x},\mathbf{w}) := \lim_{r \to 0^+} \mathcal{M}(r,\mathbf{x},\mathbf{w})$$

$$= \lim_{r \to 0^-} \mathcal{M}(r,\mathbf{x},\mathbf{w}) \qquad (6)$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{w} = (w_1, w_2)$.

From the definitions (5) and (6), it is easy to verify the following properties of the function $\mathcal{M}$.

Lemma 3.1. *For any* $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{w} = (w_1, w_2) \in \mathcal{R}_+^2$ *and* $r \in \mathbb{R}$, *the following statements hold.*

a) If $w_1 = w_2$ *then*

$\mathcal{M}(r, x_1, x_2, w_1, w_2) = \mathcal{M}(r, x_2, x_1, w_1, w_2)$.

b) If $x_1 = x_2$ then $\mathcal{M}(r, x_1, x_2, \mathbf{w}) = x_1 = x_2$ for any $\mathbf{w} \in \mathcal{R}_+^2$.

c) $\min\{x_1, x_2\} \le \mathcal{M}(r,\mathbf{x},\mathbf{w}) \le \max\{x_1, x_2\}$ and moreover

$$\left\{\mathcal{M}(r,\mathbf{x},\mathbf{w}) \mid \mathbf{w} \in \mathcal{R}_+^2\right\} = \left[\min\{x_1, x_2\}, \max\{x_1, x_2\}\right].$$

d) $\mathcal{M}(r, \alpha\mathbf{x}, \mathbf{w}) = \alpha\mathcal{M}(r,\mathbf{x},\mathbf{w})$ *for any* $\alpha > 0$.

In views of Lemma 3.1, we can define a generalization type of weighted Hölder means as follows.

Definition 3.2. For each $(r,\mathbf{x},\mathbf{w}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{R}_+^2$, $\mathcal{M}(r,\mathbf{x},\mathbf{w})$ is called the weighted Hölder type mean of order r of the components of $\mathbf{x}$ with respect to the weight $\mathbf{w}$ or $\mathbf{w}$-weighted Hölder type mean of order r of the components of $\mathbf{x}$.

Once $\mathbf{x}$ has positive components, it is easy to see that weighted Hölder type means boil down to ordinary weighted Hölder means considered in (Bullen, 2003). In the sequel, we will define and study several kinds of generalized convex functions based on these means. Before elaborating on the concepts of these functions, we further provide some properties of weighted Hölder type means defined by (5) and (6) which will be used later.

Lemma 3.3. *For each* $\mathbf{w} = (w_1, w_2) \in \mathcal{R}_+^2$ *and* $r \in \mathbb{R}$, *the function* $\mathcal{M}(r, \bullet, \bullet, w_1, w_2)$ *is increasing on* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ *in the sense that for all* $(x_1, x_2), (z_1, z_2) \in \mathcal{D}$, *if* $x_1 \le z_1$ *and* $x_2 \le z_2$ *then*

$$\mathcal{M}(r, x_1, x_2, \mathbf{w}) \le \mathcal{M}(r, z_1, z_2, \mathbf{w}) \qquad (7)$$

where

$$\mathcal{D}_1 = \{(x_1, x_2) \mid x_1 \ge 0, x_2 \ge 0\}$$

and

$$\mathcal{D}_2 = \{(x_1, x_2) \mid x_1 \le 0, x_2 \le 0\}.$$

Proof. Let $(x_1, x_2), (z_1, z_2) \in \mathcal{D}$ be arbitrary such that $x_1 \le z_1$ and $x_2 \le z_2$. We consider two possible cases on (x_1, x_2).

The first one is that $(x_1, x_2) \in \mathcal{D}_1$. If $x_1 > 0$ and $x_2 > 0$ then $z_1 > 0$ and $z_2 > 0$. Moreover, it follows that $x_1^r \le z_1^r$, $x_2^r \le z_2^r$ if $r > 0$, and that $x_1^r \ge z_1^r$, $x_2^r \ge z_2^r$ if $r < 0$. In both cases, one can verify that

$$\mathcal{M}(r, x_1, x_2, \mathbf{w}) = (w_1 x_1^r + w_2 x_2^r)^{\frac{1}{r}}$$

$$\le (w_1 z_1^r + w_2 z_2^r)^{\frac{1}{r}} = \mathcal{M}(r, z_1, z_2, \mathbf{w}). \qquad (8)$$

By taking the limit of (8) as $x_1 \to 0^+$, or $x_2 \to 0^+$, or $r \to 0$, we will come up the conclusion that $\mathcal{M}(r, x_1, x_2, \mathbf{w}) \le \mathcal{M}(r, z_1, z_2, \mathbf{w})$ for all $r \in \mathbb{R}$, $0 \le x_1 \le z_1$ and $0 \le x_2 \le z_2$.

The second case is that $(x_1, x_2) \in \mathcal{D}_2$. If $(z_1, z_2) \in \mathcal{D}_1$ then $\mathcal{M}(r, x_1, x_2, \mathbf{w}) \le 0 \le \mathcal{M}(r, z_1, z_2, \mathbf{w})$ due to the Lemma 3.1. If $z_1 < 0$ and $z_2 < 0$ then $x_1 < 0$ and $x_2 < 0$. Moreover, $0 < -z_1 \le -x_1, 0 < -z_2 \le -x_2$. It follows that $(-z_1)^r \le (-x_1)^r$ and $(-z_2)^r \le (-x_2)^r$ if $r > 0$ and that $(-z_1)^r \ge (-x_1)^r$ and $(-z_2)^r \ge (-x_2)^r$ if $r < 0$. In both cases, one can verify that

$$\mathcal{M}(r, x_1, x_2, \mathbf{w}) = -(w_1(-x_1)^r + w_2(-x_2)^r)^{\frac{1}{r}}$$

$$\le -(w_1(-z_1)^r + w_2(-z_2)^r)^{\frac{1}{r}} \qquad (9)$$

$$= \mathcal{M}(r, z_1, z_2, \mathbf{w}).$$

By taking the limit of (9) as $z_1 \to 0^-$, or $z_2 \to 0^-$, or $r \to 0$, we will come to the conclusion that $\mathcal{M}(r, x_1, x_2, \mathbf{w}) \le \mathcal{M}(r, z_1, z_2, \mathbf{w})$ for all $r \in \mathbb{R}$ and $x_1 \le z_1 \le 0$, $x_2 \le z_2 \le 0$.

Remark 3.4. In general, there exist $\mathbf{w} = (w_1, w_2) \in \mathcal{R}_+^2$ and $r \in \mathbb{R}$ so that the function $\mathcal{M}(r, \bullet, \bullet, w_1, w_2)$ is not increasing on $\mathbb{R}^2$. The following is a counter example:

Example 3.5. By taking $\mathbf{w} = (\frac{2}{3}, \frac{1}{3})$ and $r = 2$, it is easy to show that the function $\mathcal{M}(2, \bullet, \mathbf{w})$ is not increasing on $\mathbb{R}^2$ in the sense of Lemma 3.3. Indeed, we only need to consider $\mathbf{x} = (x_1, x_2), \mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$ such that $x_1 \le z_1 < 0 < x_2 \le z_2$ and $-z_1 / z_2 > 2.$ Then $\mathcal{M}(2, \mathbf{x}, \mathbf{w}) \ge \mathcal{M}(2, \mathbf{z}, \mathbf{w})$.

For fixed $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{w} = (w_1, w_2) \in \mathcal{R}_+^2$, $\mathcal{M}(r, \mathbf{x}, \mathbf{w})$ is a function in r defined on $\mathbb{R}$. This function is constant if ($w_1 = 1$ and $w_2 = 0$) or ($w_2 = 1$ and $w_1 = 0$), or $x_1 = x_2$. In the remaining cases, the following lemma provides some information on the behavior of the function.

Lemma 3.6. *For the weighted Hölder type means funtion* $\mathcal{M}(r, \mathbf{x}, \mathbf{w})$ *defined in Definition 3.2, the following statements hold.*

a) If $(x_1 > 0$ *and* $x_2 > 0)$ *or* $(\min\{x_1, x_2\} = 0$ *and* $\max\{x_1, x_2\} > 0)$ *then* $\mathcal{M}(r, \mathbf{x}, \mathbf{w})$ *is increasing in* r *on* $(-\infty, \infty)$,

b) If $(x_1 < 0$ *and* $x_2 < 0)$ *or* $(\min\{x_1, x_2\} < 0$ *and* $\max\{x_1, x_2\} = 0)$ *then* $\mathcal{M}(r, \mathbf{x}, \mathbf{w})$ *is decreasing in* r *on* $(-\infty, \infty)$.

Proof.

a) If $x_1 > 0$ and $x_2 > 0$, $\mathcal{M}(r, \mathbf{x}, \mathbf{w})$ is increasing in r due to (Mitrinović & Vasić, 1970, Theorem 1, p. 76). If $x_1 = 0$ and $x_2 > 0$. By the definition, one has

$$\mathcal{M}(r, 0, x_2, \mathbf{w}) = \lim_{\tilde{x}_1 \to 0^+} \mathcal{M}(r, \tilde{x}_1, x_2, \mathbf{w}).$$

Since $\mathcal{M}(r_1, \tilde{x}_1, x_2, \mathbf{w}) \le \mathcal{M}(r_2, \tilde{x}_1, x_2, \mathbf{w})$ for all $r_1 < r_2$, one takes the limit as $\tilde{x}_1 \to 0^+$ to obtain

$$\mathcal{M}(r_1, 0, x_2, \mathbf{w}) = \lim_{\tilde{x}_1 \to 0^+} \mathcal{M}(r_1, \tilde{x}_1, x_2, \mathbf{w})$$

$$\le \lim_{\tilde{x}_1 \to 0^+} \mathcal{M}(r_2, \tilde{x}_1, x_2, \mathbf{w}) = \mathcal{M}(r_1, 0, x_2, \mathbf{w})$$

for all $r_1 < r_2$. Therefore, $\mathcal{M}(r, 0, x_2, \mathbf{w})$ is increasing in r on $(-\infty, \infty)$ if $x_2 > 0$.

By a similar argument, one can show that $\mathcal{M}(r, x_1, 0, \mathbf{w})$ is increasing in r on $(-\infty, \infty)$ if $x_1 > 0$.

b) Applying this result to the case that $x_1 < 0$ and $x_2 < 0$, one gets that $\mathcal{M}(\cdot, -\mathbf{x}, \mathbf{w})$ is increasing in r on $(-\infty, \infty)$. Hence, $\mathcal{M}(\cdot, \mathbf{x}, \mathbf{w}) = -\mathcal{M}(\cdot, -\mathbf{x}, \mathbf{w})$ is decreasing on $(-\infty, \infty)$.

If $x_1 < 0$ and $x_2 = 0$. By the definition, $\mathcal{M}(r, x_1, 0, \mathbf{w}) = \lim_{\tilde{x}_2 \to 0^-} \mathcal{M}(r, x_1, \tilde{x}_2, \mathbf{w})$. Since

$$\mathcal{M}(r_1, x_1, \tilde{x}_2, \mathbf{w}) \ge \mathcal{M}(r_2, x_1, \tilde{x}_2, \mathbf{w})$$

for all $r_1 < r_2$, one takes the limit as $\tilde{x}_2 \to 0^-$ to obtain

$$\mathcal{M}(r_1, x_1, 0, \mathbf{w}) = \lim_{\tilde{x}_2 \to 0^-} \mathcal{M}(r_1, x_1, \tilde{x}_2, \mathbf{w})$$

$$\ge \lim_{\tilde{x}_2 \to 0^-} \mathcal{M}(r_2, x_1, \tilde{x}_2, \mathbf{w}) = \mathcal{M}(r_2, x_1, 0, \mathbf{w})$$

for all $r_1 < r_2$. Therefore, $\mathcal{M}(\cdot, x_1, 0, \mathbf{w})$ is decreasing in r on $(-\infty, \infty)$ if $x_1 < 0$.

By a similar argument, one can show that $\mathcal{M}(r, 0, x_2, \mathbf{w})$ is decreasing in r on $(-\infty, \infty)$ if $x_2 < 0$. □

3.2. Generalized convex functions of Hölder type

We begin this section with introducing the generalized (weighted) Hölder convex sets of $\mathbb{R}$.

Definition 3.7. Let $\mathcal{C}$ be a non-empty set of real numbers, r be a real number and let $\mathbf{w} \in \mathcal{R}_+^2$. The set $\mathcal{C}$ is said to be

a) $\mathbf{w}$-weighted Hölder convex of order r if $\mathcal{M}(r, \mathbf{x}, \mathbf{w})$ is in $\mathcal{C}$ for all $\mathbf{x} = (x_1, x_2) \in \mathcal{C} \times \mathcal{C}$;

b) weak Hölder convex of order r if there exists $\widetilde{\mathbf{w}} \in \mathcal{R}_+^2$ such that $\mathcal{C}$ is $\widetilde{\mathbf{w}}$-weighted Hölder convex of order r;

c) Hölder convex of order r if it is $\mathbf{w}$-weighted Hölder convex of order r for all $\mathbf{w} \in \mathcal{R}_+^2$.

By the definition, it is clear that the following implications hold for any weight $\mathbf{w} \in \mathcal{R}_+^2$: Hölder convex of order r $\Rightarrow$ $\mathbf{w}$-weighted Hölder convex of order r $\Rightarrow$ weak Hölder convex of order r. However, the converse does not hold. For example, one can verify that the set $\{0\} \cup [1, 2]$ is a weak Hölder convex set of order r, for any $r \in \mathbb{R}$, but it is not a Hölder convex set of order r.

The above definition pertains to the subsets of real numbers. In the framework of functions, we derive the following generalized concepts of Hölder convexity.

Definition 3.8. Let r and s be two real numbers, $\mathbf{w} = (w_1, w_2) \in \mathcal{R}_+^2$ and $\mathcal{C}$ be a non-empty subset of $\mathbb{R}$. A function $f : \mathcal{C} \to \mathbb{R}$ is said to be

a) $\mathbf{w}$-weighted Hölder type convex of order (r, s) if $\mathcal{C}$ is a $\mathbf{w}$-weighted Hölder convex set of order r and the following inequality

$$f(\mathcal{M}(r, x_1, x_2, \mathbf{w})) \le \mathcal{M}(s, f(x_1), f(x_2), \mathbf{w})$$

$$=: \mathcal{M}(s, f\mathbf{x}, \mathbf{w}) \qquad (10)$$

holds for all $x_1, x_2 \in \mathcal{C}$;

b) weak Hölder type convex of order (r, s) if there exists a weight $\widetilde{\mathbf{w}} \in \mathcal{R}_+^2$ such that \$f \$ is a $\widetilde{\mathbf{w}}$-weighted Hölder convex function of order (r, s);

c) Hölder type convex of order (r, s) if it is a $\mathbf{w}$-weighted Hölder convex function of order (r,s) for all weights $\mathbf{w}\in\mathcal{R}_+^2$.

Example 3.9. Consider the function $f(x)=x^{\alpha},\alpha\in\mathbb{R}$, $x\in(0,\infty)$. Let r and s be two non-zero real numbers. By Jensen's inequality, if $s>0$ and $0<\alpha s/r\le 1$, then

$$\begin{aligned}\left[\left(w_1x_1^r+w_2x_2^r\right)^{1/r}\right]^{\alpha}&=\left[\left(w_1x_1^r+w_2x_2^r\right)^{\alpha s/r}\right]^{1/s}\\ &\ge\left[w_1(x_1^r)^{\alpha s/r}+w_2(x_2^r)^{\alpha s/r}\right]^{1/s}\\ &=\left[w_1(x_1^{\alpha})^s+w_2(x_2^{\alpha})^s\right]^{1/s}\end{aligned}$$

for all weights $\mathbf{w}=(w_1,w_2)\in\mathcal{R}_+^2$ and for all $x_1,x_2\in(0,\infty)$. Hence, $f(x)=x^{\alpha}$ is a Hölder type concave function of order (r,s) if $s>0$ and $0<\alpha s/r\le 1$. Analogously, if $s>0$ and either $\alpha s/r\ge 1$ or $\alpha s/r<0$, then $f(x)=x^{\alpha}$ is a Hölder type convex function of order (r,s).

Example 3.10. Let $n\in\mathbb{N}$ be fixed, and consider the function $f(x)=\sum_{i=1}^{n}x^{\alpha_i}$, $\alpha_i\in\mathbb{R}$, $x\in(0,\infty)$. Let r and s be two non-zero real numbers. As shown in Example 3.9, if $s>0$ and either $\alpha_i s/r\ge 1$ or $\alpha_i s/r<0$ for all $i=1,2,\ldots,n$, then each function $f_i(x)=x^{\alpha_i}$ is a Hölder type convex function of order (r,s). Now we consider $0<s\le 1$ and either $\alpha_i s/r\ge 1$ or $\alpha_i s/r<0$ holds for all $i=1,2,\ldots,n$. By applying the well-known Minkowski inequality [18], we conclude that the function $f(x)=\sum_{i=1}^{n}x^{\alpha_i}$ is also a Hölder type convex function of order (r,s).

Remark 3.11. Suppose that $f:\mathcal{C}\to\mathbb{R}$ is a Hölder type generalized convex function of order (r,s). By Definition 3.8, the following statements hold.

1) The function $g:\mathcal{C}^r\to\mathbb{R}$, which is defined by $g(y)=f(x),y=x^r$, is an s-convex function that was considered in (Chen & Liu, 2013).

2) If $s=1$ then f is an r-convex function which was considered in (Fang & Shi, 2014). If $r=1$ then f boils down in, but more general setting, the class of s-convex functions defined in (Chen & Liu, 2013).

3) For $r=p\in\left\{\dfrac{2m+1}{2n+1}:m,n\in\mathbb{Z}_+\right\}$ and $s=1$, the class of Hölder convex functions of order $(p,1)$ coincides with the class of p-convex functions.

4) The notion of Hölder type convexity of order $(0,r)$ is an extension of the concept of geometrical r-convexity that was studied in (Xi & Qi, 2014).

5) For $r,s\in\{-1,0,1\}$, we obtain the classes of *MN*-convex functions, where M and N belong to the set of means $\{A,G,H\}$.

Proposition 3.12. Let r and s be real numbers and $f,g:\mathcal{C}\to(0,\infty)$ be given functions. Then, the following statements hold.

a) If f is ($\mathbf{w}$-weighted, weak) Hölder type convex (concave) of order (r,s) and $\lambda>0$, then so is λf.

b) If *f, g* are ($\mathbf{w}$-weighted, weak) Hölder type convex of order (r,s) and $s\le 1$, then so is $f+g$.

c) If *f, g* are ($\mathbf{w}$-weighted, weak) Hölder type concave of order (r,s) and $s\ge 1$, then so is $f+g$.

Proof. The first statement is obvious. The last two statements can be verified by using the well-known Minkowski's inequality (see, e.g. (Zhao & Cheung, 2011).

Proposition 3.13. Let $f:\mathcal{C}\subseteq(0,\infty)\to(0,\infty)$ be a given function. Then, the following statements hold.

a) If f is Hölder type convex of order (r,s) and is monotonically increasing, it is also Hölder type convex of order (r',s') for all $r'\le r$ and $s'\ge s$.

b) If f is Hölder type concave of order (r,s) and is monotonically decreasing, it is also Hölder type concave of order (r',s') for all $r'\le r$ and $s'\le s$.

Proof.

a) Let $x_1,x_2\in\mathcal{C}$ and $\mathbf{w}$ be a weight in $\mathcal{R}_+^2$. Let $r'\le r$ and $s'\ge s$. By Lemma 3.6 and the increasing monotonicity of f, one gets

$$f(\mathcal{M}(r',x_1,x_2,\mathbf{w}))\le f(\mathcal{M}(r,x_1,x_2,\mathbf{w}))$$

and

$$\mathcal{M}(s,f(x_1),f(x_2),\mathbf{w})\le\mathcal{M}(s',f(x_1),f(x_2),\mathbf{w}).$$

It is now easy to see that if f is Hölder type convex of order (r,s) then it is Hölder type convex of order (r',s').

b) Let $x_1,x_2\in\mathcal{C}$ and $\mathbf{w}$ be a normal weight in $\mathbb{R}^2$. Let $r'\le r$ and $s'\le s$. By Lemma \ ref{lm:nds1} and the decreasing monotonicity of f, one gets

$$f(\mathcal{M}(r',x_1,x_2,\mathbf{w}))\ge f(\mathcal{M}(r,x_1,x_2,\mathbf{w}))$$

and

$$\mathcal{M}(s',f(x_1),f(x_2),\mathbf{w})\le\mathcal{M}(s,f(x_1),f(x_2),\mathbf{w}).$$

It is now easy to see that if f is Hölder type concave of order (r,s) then it is Hölder type concave of order (r',s').

4. CONCLUSION

In this paper, based on a generalization of weighted Hölder means, we proposed a new notion of convex functions named Hölder type convex functions and studied some their properties.

CÁC DẠNG MỚI CỦA TRUNG BÌNH HÖLDER CÓ TRỌNG SỐ VÀ TÍNH LỒI

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TÓM TẮT

Trong bài báo này, dựa trên một khái niệm mới về trung bình Hölder có trọng số, chúng tôi đề xuất một khái niệm mới về tính lồi cho các hàm giá trị thực được gọi là hàm lồi tổng quát kiểu Hölder và nghiên cứu các tính chất của chúng.

Từ khóa: *Hàm lồi tổng quát; Trung bình Hölder có trọng số; Tính lồi Hölder.*

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